Algebraic Geometry
Lecture 2

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1 Affine Varieties

1.1 Affine Algebraic Sets

Let k be a field and $\overline{k}$ its algebraic closure.

$$\mathbb{A}^n := \{(a_1, a_2, \ldots, a_n) \mid a_1, a_2, \ldots, a_n \in \overline{k}\}.$$

$$\mathbb{A}^n(k) := \{(a_1, a_2, \ldots, a_n) \mid a_1, a_2, \ldots, a_n \in k\}.$$

Suppose from now on that k is algebraically closed, unless otherwise stated. In this context, k is often called the *ground field*.

Denote by $k[X]$ the polynomial ring in n variables, i.e. $k[X_1, X_2, \ldots, X_n]$. Let $S \subset k[X]$ be a set of polynomials. We define

$$V(S) := \{P \in \mathbb{A}^n | f(P) = 0 \text{ for every } f \in S\}.$$

Examples:

- Let $S = \{x^2 + y^2 - 1\}$. Then $V(S) \subset \mathbb{A}^2(\mathbb{R})$ is the circle centred at the origin.
- Let $S = \{x^2 + y^2 - 1, x\}$. Then $V(S) \subset \mathbb{A}^2(\mathbb{R})$ is the set $\{(\pm 1, 0)\}$.

A subset V of $\mathbb{A}^n$ which can be written as $V(S)$ for some subset of polynomials S is called an *affine algebraic set*.

Note that if $f, g \in S$ and $a \in k[X]$, then

- $(f + g)(P) = f(P) + g(P) = 0$ for all $P \in V(S)$ and
- $(af)(P) = af(P) = 0$ for all $P \in V(S)$.

So we might as well consider an ideal (S) generated by a set of polynomials S. $V(S) = V((S))$.

1.2 The Ideal of an Affine Algebraic Set

To every affine algebraic set V we can associate an ideal $I \subset k[X]$.

$$I(V) := \{f \in k[X] \mid f(P) = 0 \text{ for all } P \in V\}.$$

This is an *ideal* of the *ring* $k[X]$ because for all $f, g \in I(V)$ and all $a \in k[X]$:

- $f + g \in I(V)$.
- $af \in I(V)$.

The point of the ideal $I(V)$ is that we have $V(I(V)) = V(S)$. However, it is NOT always the case that $I(V) = (S)$.

Theorem 1.1. (Hilbert Basis Theorem) Every algebraic set can be given by a *finite* set S of polynomials.

Proof. This result comes from the original version of the Hilbert Basis Theorem: Every ideal in $k[X]$ is finitely generated. (Actually, if A is noetherian, then $A[X]$ is noetherian.) □

We would like to address the question: What is the connection between the ideals (S) and $I(V)$?

Example: Suppose $S = \{x^2\}$. Then $V(S) = \{0\}$ in $\mathbb{A}(\mathbb{R})$. But $I(V) = (x)$.

So we DON'T get a one-to-one correspondence between ideals and algebraic sets.

Theorem 1.2. (Nullstellensatz) If $f \in I(V)$, then $f^r \in S$ for some $r \in \mathbb{N}$.

Let J be an ideal. The *radical ideal* of J is the ideal

$$radJ = \{f \in k[X] | f^r \in J \text{ for some } r \in \mathbb{N}\}.$$

Nullstellensatz says $I(V(S)) = rad(S)$.

So we get a one-to-one correspondence between algebraic sets and RADICAL ideals.

1.3 Irreducibility and Varieties

We say that an algebraic set V is *reducible* if it can be expressed as a union $V = V_1 \cup V_2$, where V_1, V_2 are affine algebraic sets. If V cannot be expressed as such, then it is *irreducible*.

We call an irreducible affine algebraic set an *affine (algebraic) variety*.

Theorem 1.3. An affine algebraic set V is irreducible if and only if its ideal $I(V)$ is prime.

Definition 1.1. An ideal I is *prime* if whenever $ab \in I$, then $a \in I$ or $b \in I$.

2 Projective Varieties

Let k be a field and $\overline{k}$ its algebraic closure. Consider $\mathbb{A}^{n+1} \setminus \{0\}$ modulo the following equivalence relation: For all $x, y \in \mathbb{A}^{n+1} \setminus \{0\}$,

$$x \sim y \iff \exists \lambda \in \overline{k} \setminus \{0\} \text{ such that } y = \lambda x.$$

We call this set *n-dimensional projective space* and usually write its elements in terms of *homogeneous coordinates*:

$$\mathbb{P}^n := \{[a_0, a_1, a_2, \ldots, a_n] | a_0, a_1, a_2, \ldots, a_n \in \overline{k}\}.$$

where not all the a_i are zero, and $[a_0, a_1, a_2, \ldots, a_n] = [a_0', a_1', a_2', \ldots, a_n']$ iff there exists $\lambda \in \overline{k} \setminus \{0\}$ such that $a_i = \lambda a_i'$ for $i = 0, 1, \ldots, n$.

$$\mathbb{P}^n(k) := \{[a_0, a_1, a_2, \ldots, a_n] | a_1, a_2, \ldots, a_n \in k\}.$$

under the same conditions.

Example: $\mathbb{P}^2(\mathbb{R})$ is two-dimensional real projective space (the projective plane). A typical element is $[1, 2, 3]$ which is the same element as $[2, 4, 6]$.

We consider $\mathbb{P}^2(\mathbb{R})$ to be $\mathbb{A}^2(\mathbb{R})$ "plus some points at infinity". To see this, consider a general point $[X, Y, Z]$ with $Z \neq 0$. This is the same point as $[X/Z, Y/Z, 1]$. Every element $(x, y) \in \mathbb{A}^2(\mathbb{R})$ can be written as $[x, y, 1]$. The "points at infinity" are the points $[X, Y, Z]$ with $Z = 0$.

Suppose from now on that k is algebraically closed, unless otherwise stated.

We want to be able to define a projective algebraic set. However, the notion of "zero sets of polynomials" is not enough.

Example: The polynomial $Y - X^2$ has the zero $[1, 1]$. So it should also have the zero $[2, 2]$, but it doesn't.

Instead we consider only *homogeneous* polynomials, i.e. ones which satisfy

$$f(\lambda X_0, \lambda X_1, \ldots, \lambda X_n) = \lambda^d f(X_0, X_1, \ldots, X_n) \text{ for some } d \in \mathbb{N}.$$

Example: $3X^2Y + XZ^2 + Z^3 + Y^3$.

So we let S be a set of homogeneous polynomials. We define

$$V(S) := \{P \in \mathbb{P}^n \mid f(P) = 0 \text{ for every } f \in S\}.$$

A *projective algebraic set* is a subset V of $\mathbb{P}^n$ which can be written in the form $V(S)$ for some set of homogeneous polynomials S.

Similarly, we define the ideal of a projective algebraic set as

$$I(V) := \{f \in k[X] \mid f(P) = 0 \text{ for all } P \in V\}.$$

Because "all $P \in V$" refers to all scalar multiples, the ideal contains only homogeneous polynomials. Thus we call it a homogeneous ideal.

Again we have the Hilbert Basis Theorem and Nullstellensatz for projective algebraic sets. Also, a projective variety is an irreducible projective algebraic set (and it has a prime homogeneous ideal).

Example: We want to consider the elliptic curve $y^2 = x^3 + ax + b$ projectively. We *homogenise* the curve by setting $x = X/Z$ and $y = Y/Z$, giving us:

$$Y^2Z = X^3 + aXZ^2 + bZ^3.$$

The points at infinity are given by $Z = 0$: i.e. $[0, 1, 0]$.

3 The Zariski Topology

Let V be a set. A *topology* on V is a choice of subsets of V (which we shall call "open sets") such that:

1. $\emptyset$ is open.

2. V is open.

3. unions of open sets are open.

4. finite intersections of open sets are open.

A subset $U \subset V$ is called *closed* if $V \setminus U$ is open.

Define the Zariski topology on $\mathbb{A}^n$ (or $\mathbb{P}^n$) by saying all algebraic sets V are closed (equivalently, all complements of algebraic sets are open).

The Zariski topology on an algebraic set V is defined by calling all the algebraic subsets of V the closed sets.

A *quasi-affine variety* is an open subset of an affine variety.

A *quasi-projective variety* is an open subset of a projective variety.